

The Limits of Horn Logic Programs ¹

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Abstract: Given a sequence $\{\Pi_n\}$ of Horn logic programs, the limit Π of $\{\Pi_n\}$ is the set of the clauses such that every clause in Π belongs to almost every Π_n and every clause in infinitely many Π_n 's belongs to Π also. The limit program Π is still Horn but may be infinite. In this paper, we consider if the least Herbrand model of the limit of a given Horn logic program sequence $\{\Pi_n\}$ equals the limit of the least Herbrand models of each logic program Π_n . It is proved that this property is not true in general but holds if under an assumption which can be syntactically checked and be satisfied by a class of Horn logic programs. Thus, under this assumption we can approach the least Herbrand model of the limit Π by the sequence of the least Herbrand models of each finite program Π_n . We also prove that if a finite Horn logic program satisfies this assumption, then the least Herbrand model of this program is recursive. Finally, by use of the concept of stability from dynamical systems, we prove that this assumption is exactly a sufficient condition to guarantee the stability of fixed points for Horn logic programs.

Keywords: Logic Program, Horn Theory, Herbrand Model, Limit, Stability, Decidability.

1. Introduction

As time goes by, the knowledge of the human being is increasing exponentially. The amount of information on the Internet is doubled in several months. So it becomes more and more important to discover useful knowledge in massive information. There have been several data mining tools to find useful knowledge from very large databases. If the knowledge found by a data mining tool is taken as a theory at the current time, we assume that the theory is consistent. As the knowledge increases, the theories should also be updated. Thus we get a sequence of theories $\Pi_1, \Pi_2, \dots, \Pi_n, \dots$. This procedure may never stop, i.e. maybe there does not exist a natural number k such that $\Pi_k = \Pi_{k+1} = \dots$. For example, if we restrict the theories to Horn logic programs, then there exists some Herbrand interpretation I such that we will never find a finite program Π whose least Herbrand model $M = I$, because that the set of Herbrand interpretations is uncountable while the set of finite programs is only countable^[8]. So sometimes we need to consider some kind of limit of theories and discover what kind of knowledge is true in the limit.

Formally introducing limits of sequences of first order theories into logic and computer science, and using theory versions as approximations of some formal

¹The project is supported by the National 973 Project of China under the grant number G1999032701 and the National Science Foundation of China.

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theories in convergent infinite computations are independent contributions by Li in 1992. Li^{[5],[6],[7]} first defined the limits of first order theories, and thereon gave a formal system of the inductive logic. Given a sequence $\{\Pi_n\}$ of the first order theories Π_n 's, the limit $\Pi = \lim_{n \rightarrow \infty} \Pi_n$ is the set of the sentences such that every sentence in Π belongs to almost every Π_n , and every sentence in infinitely many Π_n 's belongs to Π also. The limit does not always exist for any sequence of the first order theories.

We shall consider the limits of the logic programs in terms of their Herbrand models, and focus on the Horn logic programs. There are two reasons for us to focus on the Horn logic programs. Firstly, the Horn logic programs are used as a main representation method of knowledge in the knowledge engineering. Every knowledge base must have the ability to reason, and the reasoning in a Horn logic program is feasible. Secondly, the Horn logic programs have the least Herbrand model^[10], since the meet of any two Herbrand models of a Horn logic program is a Herbrand model of this Horn logic program.

For a sequence $\{\Pi_n\}$ of finite Horn logic programs, if the limit Π of $\{\Pi_n\}$ exists then Π is a Horn logic program but it may be infinite. To discover what is true in Π , it is crucial to compute the least Herbrand model of Π . Then, the problem is: *How to construct the least Herbrand models of such Horn logic programs?*

We know that for every finite Π_n , the least Herbrand model can be constructed. Therefore, one may naturally wonder if the least Herbrand model of Π can be approached by the sequence of the least Herbrand models of Π_n . Let M_n and M be the least Herbrand models of Π_n and Π respectively. Namely, we hope to have

$$M = \lim_{n \rightarrow \infty} M_n. \quad (*)$$

In another form, $\mathbf{M}(\lim_{n \rightarrow \infty} \Pi_n) = \lim_{n \rightarrow \infty} \mathbf{M}(\Pi_n)$, where \mathbf{M} is taken as an operator which maps a Horn logic program to the least Herbrand model of the logic program.

In this paper, we prove that $(*)$ holds for a class of Horn logic programs. Thus, if $\{\Pi_n\}$ is a sequence of finite Horn logic programs in this class and $\Pi = \lim_{n \rightarrow \infty} \Pi_n$ exists, then we can approach the least Herbrand model of Π by the sequence of the least Herbrand models of each program Π_n .

The paper is arranged as follows. In section 2, we will give the basic definitions in logic programs, and the fixpoint semantics of Horn logic programs. In section 3, we will give two examples to show that $(*)$ is not true in general but holds for some Horn logic programs. In section 4, we will prove that $(*)$ is true under an assumption which can be syntactically checked and easily satisfied, and if a finite Horn logic program satisfies this assumption then the least Herbrand model of this program is recursive. In section 5, by use of the concept of stability from dynamical systems, we will prove that the assumption in section 4 is exactly a sufficient condition to guarantee the stability of fixed points for Horn logic programs.

Our notation follows from Dahr's book^[2]. We use Π to denote a logic pro-

gram, p, q atoms, π a clause and Θ a substitution. I, J denote subsets of Herbrand bases, or interpretations; n, m, r, s, N denote the natural numbers.

2. The basic definitions

Let the base (language) L consist of finite sets of constant symbols, variable symbols, function symbols and predicate symbols. A term or formula is *ground* if there is no variable in it. Let TE_L be the set of all terms in L , and TG_L of all the ground terms in L . A *literal* is an atom or the negation of an atom. An atom is called a *positive* literal whereas a negated atom is called a *negative* literal. A clause $\pi : p_1 \vee \cdots \vee p_r \leftarrow q_1, \cdots, q_s$ is called a *rule* if $r \geq 1$ and $s \geq 1$; a *fact* if $r \geq 1$ and $s = 0$; and a *goal* if $r = 0$ and $s \geq 1$. $p_1 \vee \cdots \vee p_r$ is called the *rule head*, denoted by $head(\pi)$, and q_1, \cdots, q_s is called the *rule body*, denoted by $body(\pi)$. Every q_i is called a *subgoal*. We assume that the clauses are closed, i.e., every clause is a closed formula with universal quantification on variables in the formula, for example, $p(x) \leftarrow q(x)$ means that $\forall x[p(x) \leftarrow q(x)]$. A *logic program* is a set of clauses in some base L .

Definition 2.1. A clause $\pi : p_1 \vee \cdots \vee p_r \leftarrow q_1, \cdots, q_s$ is called *Horn clause* if $r \leq 1$ and every subgoal is an atom. A logic program Π is *Horn* if every formula in Π is a Horn clause.

The *Herbrand universe* U_L of a logic program in L is the set of all ground terms in L . HB_L denotes the set of all the ground atoms in L , called the *Herbrand base*.

Definition 2.2. A *Herbrand interpretation* of a logic program Π is any subset I of HB . A Herbrand interpretation I of Π is a *Herbrand model* of Π if every clause is satisfied under I . A Herbrand model I of a logic program Π is called *minimal* if there exists no subset $I' \subseteq I$ that is a model for Π ; *least model* if I is the unique minimal model of Π .

Theorem 2.3. ([10]) Let Π be a Horn program and let $HM(\Pi)$ denote the set of all Herbrand models of Π . Then the model intersection property holds, i.e., $\bigcap_{W \in HM(\Pi)} W$ is a Herbrand model of Π .

Let S be a nonempty set and $\wp(S)$ be the power set of S . Then $(\wp(S), \subseteq)$ be a complete lattice. A mapping $f : \wp(S) \rightarrow \wp(S)$ is *monotonic* if for all element $a, b \in \wp(S)$, $a \subseteq b$ implies $f(a) \subseteq f(b)$; f is *finitary* if $f(\bigcup_{n=0}^{\infty} a_n) \subseteq \bigcup_{n=0}^{\infty} f(a_n)$ for every infinite sequence $a_0 \subseteq a_1 \subseteq \cdots$; f is *continuous* if it is monotonic and finitary.

Definition 2.4. A subset $a \subseteq \wp(S)$ is called a *pre-fixpoint* if $f(a) \subseteq a$; a *post-fixpoint* if $a \subseteq f(a)$; and a *fixpoint* if $a = f(a)$.

Let $f : \wp(S) \rightarrow \wp(S)$ be a monotonic mapping. Then f has a least fixpoint $lfp(f)$. Define by induction on n the following elements of $\wp(S)$:

$$\begin{aligned} f^0 &= \emptyset; \\ f^1 &= f(\emptyset) = f(f^0); \\ f^{n+1} &= f(f^n). \end{aligned}$$

Let $f^\omega = \bigcup_{n=0}^{\infty} f^n$.

Theorem 2.5(Kleene). Let $f : \wp(S) \rightarrow \wp(S)$ be a continuous mapping. Then $\text{lfp}(f) = f^\omega$.

In 1992, Li^[5] defined the limit of a theory sequence using the standard set-theoretic definition of limit. By this definition, given a sequence $\{\Pi_n\}$ of logic programs, we say that

$$\overline{\lim}_{n \rightarrow \infty} \Pi_n = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \Pi_j$$

is the upper limit of $\{\Pi_n\}$, and

$$\underline{\lim}_{n \rightarrow \infty} \Pi_n = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} \Pi_j$$

is the lower limit of $\{\Pi_n\}$. It follows from the definition of limits that given a sequence $\{\Pi_n\}$ of logic programs, a clause belongs to the upper limit iff it belongs to infinitely many Π_n 's, and a clause belongs to the lower limit iff it belongs to almost every Π_n . It is not hard to prove that $\underline{\lim}_{n \rightarrow \infty} \Pi_n \subseteq \overline{\lim}_{n \rightarrow \infty} \Pi_n$. If $\overline{\lim}_{n \rightarrow \infty} \Pi_n = \underline{\lim}_{n \rightarrow \infty} \Pi_n$ then we say that the (set-theoretic) limit of $\{\Pi_n\}$ exists and denote it by $\lim_{n \rightarrow \infty} \Pi_n$.

3. The limits of the Horn logic programs

Let Π be a logic program. We define a mapping $f_\Pi : 2^{HB} \rightarrow 2^{HB}$ by, for any $I \in 2^{HB}$,

$$f_\Pi(I) = \{ \text{head}(\pi)\Theta : \pi \in \Pi \ \& \ \exists \Theta [\text{body}(\pi)\Theta \subseteq I] \}.$$

Proposition 3.1([10]). If Π is a Horn logic program then f_Π is monotonic and finitary. Moreover, f_Π^ω is the least model of Π .

We hope to prove the following statement:

(3.1) Given a sequence $\{\Pi_n\}$ of the Horn logic programs, let M_n be the least model of Π_n . If $\Pi = \lim_{n \rightarrow \infty} \Pi_n$ exists then Π is a Horn logic program. Let M be the least model of Π , then $M = \lim_{n \rightarrow \infty} M_n$.

The Claim (3.1) does not hold for all Horn logic programs. Let us take a look at the following example.

Example 3.2. Assume that there are one function symbol f , one predicate symbol p and one constant symbol a in L . Let $\{\Pi_n\}$ be defined as follows:

$$\begin{aligned} \Pi_1 &= \{p(x) \leftarrow p(f(x)), p(f(a))\}, \\ \Pi_2 &= \{p(x) \leftarrow p(f(x)), p(f^2(a))\}, \\ &\dots \\ \Pi_n &= \{p(x) \leftarrow p(f(x)), p(f^n(a))\}, \end{aligned}$$

where $f^{(n)}(a) = f(f^{(n-1)}(a))$, $f^1(a) = f(a)$. Then we have

$$\begin{aligned} f_{\Pi_1}^\omega &= \{p(a), p(f(a))\}, \\ f_{\Pi_2}^\omega &= \{p(a), p(f(a)), p(f^2(a))\}, \\ &\dots \\ f_{\Pi_n}^\omega &= \{p(a), p(f(a)), \dots, p(f^n(a))\}, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} f_{\Pi_n}^\omega = \{p(a), p(f(a)), \dots, p(f^n(a)), \dots\}.$$

But

$$\Pi = \lim_{n \rightarrow \infty} \Pi_n = \{p(x) \leftarrow p(f(x))\},$$

and

$$f_\Pi^\omega = \emptyset.$$

Therefore, $f_\Pi^\omega \neq \lim_{n \rightarrow \infty} f_{\Pi_n}^\omega$.

For some other Horn logic programs, the Claim (3.1) is true. Let us take a look at the following example.

Example 3.3. Assume that there are one function symbol f , one predicate symbol p and one constant symbol a in L . Let $\{\Pi_n\}$ be defined as follows:

$$\begin{aligned} \Pi_1 &= \{p(f(x)) \leftarrow p(x), p(f(a))\}, \\ \Pi_2 &= \{p(f(x)) \leftarrow p(x), p(f^2(a))\}, \\ &\dots \\ \Pi_n &= \{p(f(x)) \leftarrow p(x), p(f^n(a))\}, \end{aligned}$$

where $f^n(a) = f(f^{n-1}(a))$, $f^1(a) = f(a)$. Then we have

$$\begin{aligned} f_{\Pi_1}^\omega &= \{p(f(a)), p(f^2(a)), \dots, p(f^n(a)), \dots\}, \\ f_{\Pi_2}^\omega &= \{p(f^2(a)), p(f^3(a)), \dots, p(f^n(a)), \dots\}, \\ &\dots \\ f_{\Pi_n}^\omega &= \{p(f^n(a)), p(f^{n+1}(a)), \dots\}, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} f_{\Pi_n}^\omega = \emptyset.$$

Now we have

$$\Pi = \lim_{n \rightarrow \infty} \Pi_n = \{p(f(x)) \leftarrow p(x)\},$$

and

$$f_\Pi^\omega = \emptyset.$$

Therefore, $f_\Pi^\omega = \lim_{n \rightarrow \infty} f_{\Pi_n}^\omega$.

4. The main theorems

Example 3.2 shows that the Claim (3.1) does not hold for all Horn logic programs. In general cases, we have the following property.

Lemma 4.1. For any base L , given any sequence $\{\Pi_n\}$ of the Horn logic programs in L , if $\Pi = \lim_{n \rightarrow \infty} \Pi_n$ exists then $\varinjlim_{n \rightarrow \infty} f_{\Pi_n}^\omega$ is a Herbrand model of Π . Therefore, $f_\Pi^\omega \subseteq \varinjlim_{n \rightarrow \infty} f_{\Pi_n}^\omega$.

Proof. To show that $f_\Pi^\omega \subseteq \varinjlim_{n \rightarrow \infty} f_{\Pi_n}^\omega$, let p be a ground atom $\in f_\Pi^\omega$, we prove the claim by induction on m such that $p \in f_\Pi^{m+1} - f_\Pi^m$. There are two cases:

Case 1. $p \in f_\Pi^1$. Then $[p \leftarrow] \in \Pi$. Hence, for almost every n , $[p \leftarrow] \in \Pi_n$, so $p \in f_{\Pi_n}^1 \subseteq f_{\Pi_n}^\omega$. Hence, $p \in \varinjlim_{n \rightarrow \infty} f_{\Pi_n}^\omega$.

Case 2. $p \in f_\Pi^{m+1} - f_\Pi^m$. By the definition of f_Π , there is a clause $\pi \in \Pi$ and a substitution Θ such that $body(\pi)\Theta \subseteq f_\Pi^m$ and $p = head(\pi)\Theta$. By the induction assumption, $body(\pi)\Theta \in \varinjlim_{n \rightarrow \infty} f_{\Pi_n}^\omega$, i.e., there is an N_0 such that for any $n \geq N_0$, $body(\pi)\Theta \subseteq f_{\Pi_n}^\omega$. Since $\pi \in \Pi$, then $\pi \in \Pi_n$ for almost every n . Let N_1 be the least such that for any $n \geq N_1$, $\pi \in \Pi_n$. Let $N = \max\{N_0, N_1\}$. Hence, by the same substitution Θ , $p = head(\pi)\Theta \in f_{\Pi_n}^\omega$ for any $n \geq N$. \square

Example 3.2 and lemma 4.1 indicate that to make the Claim (3.1) hold, we have to put some conditions on Horn logic programs.

Assumption 4.2. Given a Horn logic program Π , assume that every clause $\pi : p \leftarrow p_1, \dots, p_m$ in Π has the following property: for every i if t is a term in p_i , then t is also in p .

Nienhuys-Cheng^[9] used the above condition to discuss a metric on the set of Herbrand interpretations for finite Horn logic programs. As we know, in a constrained clause, it is usually assumed that every variable occurring in the body of a clause also occurs in its head. In fact, assumption 4.2 can be syntactically checked in polynomial time and easily satisfied. For example, suppose that there are one function symbol f , one predicate symbol p and one constant symbol a in the base L . It is easy to verify that there are infinitely many clauses satisfying this assumption, e.g. $\pi_k = p(f^k(x)) \leftarrow p(x)$, where $k = 1, 2, \dots$. Based on this fact, we can conclude that there is an uncountable number of infinite Horn logic programs satisfying assumption 4.2.

By the definition of the limits and the assumption that every Π_n is Horn, it is easily verified that $\Pi = \lim_{n \rightarrow \infty} \Pi_n$ is Horn. Under assumption 4.2, we shall prove the following theorem.

Theorem 4.3. Let L be a base consisting of infinitely many constant symbols, function symbols and finitely many predicate symbols. Given a sequence $\{\Pi_n\}$ of Horn logic programs in L , if $\Pi = \lim_{n \rightarrow \infty} \Pi_n$ exists and there is an N such that Π_n satisfies assumption 4.2 for every $n \geq N$, then $\lim_{n \rightarrow \infty} f_{\Pi_n}^\omega$ exists and $f_\Pi^\omega = \lim_{n \rightarrow \infty} f_{\Pi_n}^\omega$.

Proof. We first prove that $\overline{\lim_{n \rightarrow \infty} f_{\Pi_n}^\omega} \subseteq f_\Pi^\omega$. Assume that $p \in \overline{\lim_{n \rightarrow \infty} f_{\Pi_n}^\omega}$, then there is an infinite sequence $n_i, i = 1, 2, \dots$, such that for every $n_i, p \in f_{\Pi_{n_i}}^\omega$. Then by the definition, for every n_i , there is a proof of p from the facts of Π_{n_i} . We define a proof tree T of p as follows such that every node α of T is associated

with a set of atoms, denoted by $P(\alpha)$. Let λ be the root of T , assume that $P(\lambda) = \{p\}$. For every clause $\pi \in \bigcup_{n=N}^{\infty} \Pi_n$ with head p , there is a node $\alpha \in T$ such that α is a child of λ and $P(\alpha) = \text{body}(\pi)$. For every node $\alpha \in T$, if there is a clause $\pi' \in \bigcup_{n=N}^{\infty} \Pi_n$ such that $\text{head}(\pi') \in P(\alpha)$, there is a child β of α such that $P(\beta) = \text{body}(\pi')$ (if π is a fact then $P(\beta) = \emptyset$).

We construct a subtree T' of T as follows: let $\lambda \in T'$, and let α be a child of λ such that $p \leftarrow P(\alpha)$ is used to deduce p , by the assumption, such an α does exist, then set $\alpha \in T'$; and for any node $\alpha' \in T'$ and $p' \in P(\alpha')$, let β be the child of α' such that $p' \leftarrow P(\beta)$ is used to deduce p' , then set $\beta \in T'$.

We now prove that T has only finitely many such subtrees. To do this, we only need to show that T is finite. Every ground atom has a tree structure and the depth of the tree can be considered as the *level* of the atom^[9]. By assumption 4.2, the level of an atom in T is not smaller than that of any atom in its children. Hence, the level of every atom in T is not greater than that of p , i.e. the levels of atoms in T are finite. Also by assumption 4.2, every function symbol or constant symbol appearing in the nodes of T also appears in p . Note that there are only finitely many predicate symbols. Thus T is finite and so we are done.

It follows from the above analysis that there is a subtree T' of T such that T' belongs to infinitely many Π_n 's. By the definition of the limits, we have

$$T' \subseteq \lim_{n \rightarrow \infty} \Pi_n.$$

There is an N_0 and a K such that for every $n \geq N_0$, $T' \in f_{\Pi_n}^K$. So $p \in f_{\Pi}^K$, i.e., $p \in f_{\Pi}^{\omega}$. Note that $p \in \overline{\lim}_{n \rightarrow \infty} f_{\Pi_n}^{\omega}$. Thus $\overline{\lim}_{n \rightarrow \infty} f_{\Pi_n}^{\omega} \subseteq f_{\Pi}^{\omega}$. By lemma 4.1, $f_{\Pi}^{\omega} \subseteq \underline{\lim}_{n \rightarrow \infty} f_{\Pi_n}^{\omega}$. Recall that $\underline{\lim}_{n \rightarrow \infty} f_{\Pi_n}^{\omega} \subseteq \overline{\lim}_{n \rightarrow \infty} f_{\Pi_n}^{\omega}$. So, we get

$$\underline{\lim}_{n \rightarrow \infty} f_{\Pi_n}^{\omega} = \overline{\lim}_{n \rightarrow \infty} f_{\Pi_n}^{\omega} = f_{\Pi}^{\omega}.$$

This completes the proof. \square

The following discussion will point out an interesting decidability result on the least Herbrand models related to the above main theorem. As we know, given a finite Horn logic program, the least Herbrand model of this program is in general non-recursive^[1]. In what follows, we will prove that for finite Horn logic programs satisfying assumption 4.2, their least Herbrand models are recursive. More precisely, we have the following theorem.

Theorem 4.4. Given a finite Horn logic program Π , if Π satisfies assumption 4.2, then the least Herbrand model of Π is recursive.

Proof. For any ground atom p , using the clauses in Π , we can construct a tree T with p as the root node by the same method as in the proof of theorem 4.3. As previously shown, if Π satisfies assumption 4.2, then T is finite. Note that T contains all the possible proof trees for p from Π . So we can decide in finite time whether p is provable from Π . Equivalently, it is decidable whether p is in the least Herbrand model of Π . This completes the proof. \square

5. The stability of fixed points for Horn logic programs

For a Horn logic program Π , the least Herbrand model of this program is also a fixed point of the mapping f_Π . In the study of dynamical systems, the concept of stability is of great importance to characterize the properties of fixed points. Intuitively, the stability of a fixed point of a differential equation means that if the initial solution is sufficiently close to the fixed point then the solution will remain close to the fixed point thereafter^[3]. There are many ways to formalize this concept mathematically which are similar but not the same. A classical concept for the stability of fixed points defined by Lyapounov is as follows^[3]:

A fixed point X of $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is *stable* if for all $\epsilon > 0$ there exists $\delta(\epsilon)$ such that

$$|F^n(x_0) - X| < \epsilon \quad \text{for } n = 1, 2, \dots,$$

for all x_0 such that $|x_0 - X| < \delta(\epsilon)$, where $x_n = F^n(x_0)$ is defined iteratively by $x_{n+1} = F(x_n)$.

To discuss the stability of fixed points for Horn logic programs, it is essential to define a distance between Herbrand interpretations. Fitting^[4] was the first to introduce metric methods to logic programming. [4] defined a metric on the set of Herbrand interpretations by level mappings and studied the fixed point semantics for some logic programs using the Banach contraction theorem. Nienhuys-Cheng^[9] used the depth of an expression tree for the level mapping and discussed this mapping and the induced metric.

Definition 5.1. ([9]) Let HB be the Herbrand base of a logical language. A level mapping is a function $\| : HB \rightarrow \mathbb{N}$ (natural numbers). We use $|A|$ to denote $\|(A)$, i.e. the image of A . It is called the *level* of A .

Given a level mapping $\|$ we can define a metric on the set of all Herbrand interpretations as follows: $d(I, I) = 0$ for every interpretation I . If $I \neq J$, then $d(I, J) = 1/2^n$ if I and J differ on some ground atom of level n , but agree on all ground atoms of lower level^[4], i.e. $A \in I \cap J$ for all $|A| \leq n - 1 \wedge A \in I \cup J$ and there is an $A \in I \Delta J$ such that $|A| = n$.

Theorem 5.2. ([4]) The function d defined above is a metric on the set of all Herbrand interpretations 2^{HB} .

Lemma 5.3. ([4]) The metric space $(2^{HB}, d)$ is complete.

Definition 5.4. ([9]) Every ground atom has a tree structure and the depth can be considered as the level of the atom. Thus we have a level mapping $\| : HB \rightarrow \mathbb{N}$. Let d be the distance on the set of Herbrand interpretations induced by this level mapping.

For a Horn logic program Π , the mapping f_Π is from the set of Herbrand interpretations to the set of Herbrand interpretations. In this paper, we will use the above metric to study the fixed points of Horn logic programs. First, if a Horn logic program Π satisfies assumption 4.2, then the mapping f_Π has the following property.

Lemma 5.5. ([9]) Given a Horn logic program Π , if Π satisfies assumption 4.2, then $d(f_\Pi(I), f_\Pi(J)) \leq d(I, J)$ for all $I, J \in 2^{HB}$.

Given a Horn logic program Π , if there is a constant $c : 0 \leq c < 1$ such that $d(f_\Pi(I), f_\Pi(J)) < c \cdot d(I, J)$ for all $I, J \in 2^{HB}$, then the mapping f_Π is called a

contraction. In such a case, we know, by the Banach contraction theorem, that f_Π has a unique fixed point. For a Horn logic program Π satisfying assumption 4.2, in many cases the mapping f_Π may be a contraction, e.g. $\Pi = \{p(f(x)) \leftarrow p(x), p(f(a))\}$, but not in general, e.g. $\Pi = \{p(x) \leftarrow p(x), p(f(a))\}$ which has more than one fixed point such as $\{p(f(a))\}$ and $\{p(a), p(f(a))\}$.

We can now define and discuss the stability of fixed points for Horn logic programs.

Definition 5.6. For a Horn logic program Π , a fixed point J of $f_\Pi : 2^{HB} \rightarrow 2^{HB}$ is *stable* if for all $\epsilon > 0$ there exists $\delta(\epsilon)$ such that

$$d(f_\Pi^n(I), J) < \epsilon \quad \text{for } n = 1, 2, \dots,$$

for all interpretations I such that $d(I, J) < \delta(\epsilon)$.

From the above definition, we can prove that the fixed points of Horn logic programs satisfying assumption 4.2 are stable. That is to say, we have the following theorem.

Theorem 5.7. Given a Horn logic program Π , if Π satisfies assumption 4.2, and J is a fixed point of f_Π , then J is a stable fixed point.

Proof. To prove that J is a stable fixed point, by definition 5.6, we have to show that for any $\epsilon > 0$ there exists $\delta(\epsilon)$ such that

$$d(f_\Pi^n(I), J) < \epsilon \quad \text{for } n = 1, 2, \dots,$$

for all interpretations I such that $d(I, J) < \delta(\epsilon)$.

Let $\delta(\epsilon) = \epsilon$. We can now prove the above inequality by induction on n . When $n = 0$, it is trivial to see that the inequality holds. Assume that for $n = k$, the inequality also holds, i.e.

$$d(f_\Pi^k(I), J) < \epsilon.$$

When $n = k + 1$, applying lemma 5.2 yields

$$d(f_\Pi^{k+1}(I), J) = d(f_\Pi^{k+1}(I), f_\Pi^{k+1}(J)) \leq d(f_\Pi^k(I), J) < \epsilon,$$

as required. □

By theorem 5.7, it is easy to see that in example 3.3, the least Herbrand model $M = \emptyset$ of the program $\Pi = \{p(f(x)) \leftarrow p(x)\}$ is a stable fixed point. But in example 3.2 where assumption 4.2 is not satisfied, we can prove by contradiction that the least Herbrand model $M = \emptyset$ of the program $\Pi = \{p(x) \leftarrow p(f(x))\}$ is not a stable fixed point. Suppose that $M = \emptyset$ is a stable fixed point. By definition 5.6, for $\epsilon = 1/8$, there exists a constant $\delta > 0$ such that

$$d(f_\Pi^n(I), M) < \frac{1}{8} \quad \text{for } n = 1, 2, \dots,$$

for all interpretations I such that $d(I, M) < \delta$. Let $I_0 = \{p(f^k(a))\}$ where $k \in \mathbb{N}$ and $2^k > 1/\delta$. Then we have

$$d(I_0, M) = \frac{1}{2^{k+2}} < \frac{1}{2^k} < \delta,$$

and $f_{\Pi}^k(I_0) = \{p(a)\}$. So $d(f_{\Pi}^k(I_0), M) = 1/4$, which contradicts the assumption that $d(f_{\Pi}^n(I_0), M) < 1/8$ for $n = 1, 2, \dots$, and thus completes the proof.

Theorems 4.3, 5.7 and examples 3.2, 3.3 show that there seems to exist some connections between the limit of Horn logic programs and the stability of fixed points for Horn logic programs. As mentioned above, a stable fixed point is one where a small perturbation will not grow but remain close to the fixed point even after an infinite number of iterations. So from this point of view we can understand that if the least Herbrand model of a Horn logic program is a stable fixed point, then this model may be approached by a sequence of least Herbrand models.

6. Conclusion

Assume that the base L contains infinitely many constant symbols, function symbols and finitely many predicate symbols. Given a sequence $\{\Pi_n\}$ of Horn logic programs, suppose that the limit Π of $\{\Pi_n\}$ exists, we studied if the least Herbrand model of Π is equal to the limit of the least Herbrand models of each Π_n . It was shown by some examples that in general, this property is not true. Further studies proved that the property holds if Horn logic programs satisfy an assumption that can be syntactically checked and easily satisfied. This result implies that for an infinite Horn logic program satisfying the assumption, the least Herbrand model of this program can be approached by a sequence of least Herbrand models of finite programs. We also discussed the decidability of least Herbrand models for Horn logic programs satisfying the assumption. It is proved that the least Herbrand models of these programs are recursive. More interestingly, using the concept of stability from dynamical systems, we proved that the assumption is exactly a sufficient condition to ensure the stability of fixed points for Horn logic programs. Finally, we hope that more general conditions could be found for the property to hold and the main ideas and results of this paper could be extended to more general logic programs in the future.

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